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DISCRETE MATHEMATICS

Note

Five mutually orthogonal Latin squares of orders 24 and 40

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Abstract

Let $N(n)$ denote the maximum number of mutually orthogonal Latin squares of order n. It is proved that $N(24)$ and $N(40) \geqslant 5$.

1. Introduction

Denote by $N(n)$ the maximum number of mutually orthogonal Latin squares (MOLS) of order n.

In 1979 Brouwer [1] published a table of the best known lower bounds for $N(n)$, $n < 10000$. The table has been improved many times since then. In 1987 Roth and Peters [3] proved that $N(24) \geqslant 4$. As early as in 1922 Mac Neish [2] proved a theorem which implies $N(40) \geqslant 4$. In this paper we prove that $N(24)$ and $N(40) \geqslant 5$. Additionally, we show a new construction of four MOLS of order 20. The construction differs from that given by Todorov [4].

2. Definitions

Let G be an abelian group of order n. An (r, n)-difference matrix $D = (d_{ij})$ over G is an $r \times n$ matrix satisfying the condition

$$\{d_{ik} - d_{jk}\colon k = 0, 1, \ldots, n-1\} = G \quad \text{for every } i, j = 0, 1, \ldots, r-1$$

such that $i \neq j$.

It is well known that the existence of an (n, r)-difference matrix implies $N(n) \geqslant r-1$.

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A square matrix $H=(h_{ij})$ of order n whose elements are $+1$ or -1 is called a Hadamard matrix if its rows are mutually orthogonal i.e.

$$\sum_{k=0}^{n-1} h_{ik}h_{jk}=0 \quad \text{for } i,j=0,1,\ldots,n-1,\ i\neq j.$$

It is well known that Hadamard matrices can exist if $n(\bmod 4)\equiv 0$ or $n=2$.

Let $D=(d_{ij})$ be an (r,n)-difference matrix over $G=Z_s\oplus Z_2\oplus Z_2$, $n=4s$. Without loss of generality we can put $d_{0j}=0, d_{1j}=j$ for $j=0,1,\ldots,n-1$. For convenience, the element (a,b,c) of G will be denoted by the integer $4a+2b+c$. We define the matrix $H^D=(h_{ij})$ as follows: $h_{ij}=+1$ if d_{ij} is an even integer, $h_{ij}=-1$ if d_{ij} is an odd integer. It is easy to see that the rows of H^D are mutually orthogonal.

Partition the set $B_i=\{0,1,\ldots,n-1\}$, $i=1,\ldots,5$, into two-element subsets as follows:

$$i=1:\ \{0,1\},\{2,3\},\ldots,\{4s-2,4s-3\}.$$
$$i=2:\ \{0,2s\},\{1,2s+1\},\ldots,\{2s-1,4s-1\}.$$
$$i=3:\ \{0,2\},\{4,6\},\ldots,\{4s-4,4s-2\},$$
$$\{1,3\},\{5,7\},\ldots,\{4s-3,4s-1\}.$$
$$i=4:\ \{0,3\},\{4,7\},\ldots,\{4s-4,4s-1\},$$
$$\{1,2\},\{5,6\},\ldots,\{4s-3,4s-2\}.$$
$$i=5:\ \{0,2s+1\},\{2,2s+3\},\ldots,\{2s-2,4s-1\},$$
$$\{1,2s\},\{3,2s+2\},\ldots,\{2s-1,4s-2\}.$$

The set of all two-element subsets partitioning B_i will be denoted by C_i, $i=1,\ldots,5$.

We shall construct $(6,n)$-difference matrices D for $n=24$ and 40 with elements d_{ij} satisfying the condition

$$\{d_{ij},d_{i,j+1}\}\in C_i \quad \text{for } i=1,\ldots,5,\ j=0,2,\ldots,n-2.$$

Hence, if d_{ij} is fixed, then $d_{i,j+1}$ is determined uniquely.

3. Five MOLS of order 24

Let

$$A=\begin{pmatrix}
+&+\\
+&-&+&-&+&-&+&-&+&-&+&-&+&-&+&-&+&-&+&-&+&-&+&-\\
+&+&+&+&+&+&+&+&+&+&+&+&-&-&-&-&-&-&-&-&-&-&-&-\\
+&+&-&-&+&+&-&-&+&+&-&-&+&+&-&-&+&+&-&-&+&+&-&-\\
+&-&-&+&+&-&-&+&+&-&-&+&+&-&-&+&+&-&-&+&+&-&-&+\\
+&-&+&-&-&+&-&+&+&-&+&-&-&+&-&+&+&-&+&-&-&+&-&+
\end{pmatrix}$$

where + and − denote +1 and −1, respectively. The rows of A are rows of a Hadamard matrix of order 24. Our search was restricted to (6, 24)-difference matrices such that $H^D = A$.

We wrote a computer program that found over one hundred thousand (6, 24)-difference matrices over $Z_6 \oplus Z_2 \oplus Z_2$. When presenting (6, 24)-difference matrices D we write down the elements d_{ij} for $j = 0, 2, \ldots, n-2$ and $i = 1, 2, \ldots, 5$ as d_{ij} determines $d_{i,j+1}$ if j is an even integer. The row consisting of zeros is omitted here.

$$\left(\begin{array}{cccccccccccc}
0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 \\
0 & 6 & 2 & 20 & 16 & 22 & 7 & 1 & 21 & 3 & 23 & 17 \\
0 & 5 & 10 & 13 & 20 & 19 & 6 & 3 & 14 & 9 & 16 & 23 \\
0 & 9 & 16 & 21 & 12 & 7 & 20 & 13 & 8 & 5 & 18 & 1 \\
0 & 20 & 23 & 5 & 4 & 14 & 11 & 15 & 18 & 8 & 1 & 19
\end{array}\right)$$

The same applies to (6, 40)-difference matrices.

4. Five MOLS of order 40

Following a similar way and imposing some additional conditions on D we found by a nonexhaustive computer search a number of (6, 40)-difference matrices over $G = Z_{10} \oplus Z_2 \oplus Z_2$. One of them is presented here.

$$\left(\begin{array}{cccccccccccccccccccc}
0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 \\
0 & 4 & 28 & 12 & 36 & 2 & 26 & 10 & 34 & 38 & 5 & 1 & 13 & 29 & 3 & 37 & 11 & 27 & 39 & 35 \\
0 & 5 & 16 & 21 & 14 & 37 & 8 & 33 & 24 & 31 & 6 & 3 & 22 & 19 & 38 & 13 & 34 & 11 & 28 & 27 \\
0 & 11 & 2 & 37 & 34 & 31 & 18 & 25 & 32 & 5 & 30 & 21 & 16 & 23 & 10 & 15 & 4 & 39 & 24 & 13 \\
0 & 14 & 13 & 5 & 4 & 16 & 17 & 9 & 10 & 20 & 33 & 23 & 26 & 34 & 39 & 27 & 30 & 38 & 3 & 29
\end{array}\right)$$

Thus we proved the existence of five MOLS of orders 24 and 40. The method works for $n = 8p$, where p is a prime integer, but it does not lead to improvement of $N(n)$ if $p > 5$.

5. Four MOLS of order 20

Let $G = Z_{10} \oplus Z_2$ and denote the element (a, b) of G by $2a + b$. Imposing some restrictions on the second and third rows of (5, 20)-difference matrix we found over 500 constructions. One of them is given here.

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 0 & 10 & 4 & 14 & 8 & 18 & 2 & 12 & 16 & 6 & 11 & 1 & 15 & 5 & 19 & 9 & 13 & 3 & 7 & 17 \\ 0 & 11 & 17 & 6 & 12 & 3 & 9 & 18 & 14 & 5 & 1 & 10 & 16 & 7 & 13 & 2 & 8 & 19 & 15 & 4 \\ 0 & 3 & 12 & 8 & 15 & 1 & 19 & 4 & 5 & 10 & 14 & 18 & 17 & 11 & 9 & 7 & 2 & 16 & 6 & 13 \end{pmatrix}$$

As previously, the row consisting of zeros has been omitted.

References

[1] A.E. Brouwer, The number of mutually orthogonal Latin squares — a table up to order 10 000, Research Report ZW 123/79, Mathematisch Centrum, 1979.

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[3] R. Roth and M. Peters, Four pairwise orthogonal Latin squares of order 24, J. Combin. Theory Ser. A 44 (1987) 152–155.

[4] D.T. Todorov, Four mutually orthogonal Latin squares of order 20, Ars Combin. 27 (1989) 63–65.