Three Mutually Orthogonal Latin Squares of Order 14

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Let $N(n)$ denote the maximum number of mutually orthogonal Latin squares of order n. It is known that if $n \geq 7$, $n \neq 10,14$ then $N(n) \geq 3$ [1,2]. The existence of a set of s mutually orthogonal Latin squares of order n is equivalent to the existence of an orthogonal array $OA(n,s+2)$ [3]. The orthogonal array $OA(n,s)$ is defined as a $(s \times n^2)$ matrix over $\{1,\ldots,n\}$, the rows of which are mutually orthogonal, i.e. if

$$r_1 = (x_1,\ldots,x_{n^2}),\ \ r_2 = (y_1,\ldots,y_{n^2})$$

are rows from $OA(n,s)$ then every pair (f,g), $1 \leq f,g \leq n$ occurs exactly once in the set of all ordered pairs (x_i,y_i) $i = 1,\ldots,n^2$. Below, a construction of $OA(14,5)$ is given.

We add a point ∞ to Z_{13} (the cycle of residues mod 13), and define for every $a \in Z_{13}$

$$a + \infty = \infty + a = a \cdot \infty = \infty \cdot a = \infty$$

(the addition and multiplication are assumed to be in Z_{13}).

Let $A = \|a_{ij}\|$ $i = 1,\ldots,5$, $j = 1,\ldots,15$ be a matrix over $Z_{13} \cup \{\infty\}$. Let $r_1 = (x_1,\ldots,x_{15})$, $r_2 = (y_1,\ldots,y_{15})$ be rows of A. We say that r_1,r_2 are orthogonal if

(1) $(x_i,y_i) \neq (\infty,\infty)$, $i = 1,\ldots,15$,

(2) there exist integers i_0,j_0, $1 \leq i_0 \neq j_0 \leq 15$ such that $x_{i_0} = \infty$, $y_{j_0} = \infty$,

(3) for every $a \in Z_{13}$ there exists a pair (x_i,y_i) with $x_i - y_i = a$.

The matrix A is said to be a *DS*-matrix if its rows are mutually orthogonal.

For $b \in Z_{13}$ define $A + b = \|a_{ij}+b\|$, $i = 1,\ldots,5$, $j = 1,\ldots,15$. Obviously the matrix

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$$\left\| \begin{array}{c|c|c|c|c} \infty & & & & \\ \infty & & & & \\ \infty & A & A+1 & \cdots & A+12 \\ \infty & & & & \\ \infty & & & & \end{array} \right\|$$

is an $OA(14,5)$ orthogonal array over $Z_{13} \cup \{\infty\}$.

Clearly we can add an integer modulo 13 to any row, as well as to any column of A without losing the DS-property, so we shall assume that the first row of A is $(\infty, a, a, \ldots, a)$, and the first column is (∞, a, a, a, a) for some $a \in Z_{13}$. The choice of a is not essential, so we remove the first column, as well as the first row of A, denoting the remaining (4×14) matrix by AR. Clearly:

(i) every row of AR is a permutation of $Z_{13} \cup \{\infty\}$,

(ii) every column of AR contains different elements from $Z_{13} \cup \{\infty\}$,

(iii) if $r_1 = (x_1, \ldots, x_{14})$, $r_2 = (y_1, \ldots, y_{14})$ are rows from AR, then for every $a \in Z_{13} \setminus \{0\}$ there exists exactly one pair (x_i, y_i) with $a = x_i - y_i$.

Now suppose that

$$r_1 = (x_1, \ldots, x_m, \infty, x_{m+2}, \ldots, x_{s-1}, \alpha, x_{s+1}, \ldots, x_{12}),$$

$$r_2 = (y_1, \ldots, y_m, \beta, y_{m+2}, \ldots, y_{s-1}, \infty, y_{s+1}, \ldots, y_{12})$$

are rows from AR. Since A is a DS-matrix, then

$$(x_1 - y_1) + \ldots + (x_m - y_m) + (x_{m+2} - y_{m+2}) + \ldots +$$
$$(x_{s-1} - y_{s-1}) + (x_{s+1} - y_{s+1}) + \ldots + (x_{12} - y_{12}) = 0.$$

Therefore

$$\sum_{i=1}^{12} x_i = \sum_{i=1}^{12} y_i.$$

On the other hand

$$\sum_{i=1}^{12} x_i = -\alpha, \quad \sum_{i=1}^{12} y_i = -\beta$$

due to (i). This yields $\alpha = \beta$.

Since $A + b$, bA are DS-matrices for every $b \in Z_{13} \setminus \{0\}$ then we can conclude that 4 columns of AR represent a symmetric matrix of the type:

$$AR = \begin{Vmatrix} \infty & 0 & 1 & \alpha_1 \\ 0 & \infty & \alpha_2 & \alpha_3 \\ 1 & \alpha_2 & \infty & \alpha_4 \\ \alpha_1 & \alpha_3 & \alpha_4 & \infty \end{Vmatrix}$$

If checked, 375 different matrices of this kind can be established (note that the columns and rows of AR can be reordered).

Exactly three of them were extended to AR matrices, which was done on IBM 4331.

$$\begin{Vmatrix} \infty & 0 & 1 & 3 & 2 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & \infty & 2 & 12 & 10 & 7 & 9 & 5 & 4 & 1 & 11 & 8 & 3 & 6 \\ 1 & 2 & \infty & 9 & 5 & 3 & 12 & 7 & 11 & 0 & 4 & 6 & 8 & 10 \\ 3 & 12 & 9 & \infty & 6 & 2 & 7 & 11 & 1 & 5 & 10 & 0 & 4 & 8 \end{Vmatrix}$$

$$\begin{Vmatrix} \infty & 0 & 1 & 3 & 2 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & \infty & 7 & 4 & 5 & 8 & 10 & 1 & 3 & 2 & 6 & 12 & 9 & 11 \\ 1 & 7 & \infty & 12 & 4 & 10 & 3 & 5 & 8 & 11 & 0 & 2 & 6 & 9 \\ 3 & 4 & 12 & \infty & 9 & 1 & 6 & 11 & 2 & 10 & 8 & 0 & 7 & 5 \end{Vmatrix}$$

$$\begin{Vmatrix} \infty & 0 & 1 & 3 & 2 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & \infty & 10 & 6 & 9 & 12 & 3 & 7 & 11 & 5 & 8 & 2 & 4 & 1 \\ 1 & 10 & \infty & 2 & 7 & 6 & 11 & 0 & 8 & 4 & 12 & 5 & 9 & 3 \\ 3 & 6 & 2 & \infty & 4 & 11 & 9 & 1 & 12 & 7 & 5 & 0 & 8 & 10 \end{Vmatrix}$$

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